Chapter
Applications of Integrals

Topic-1: Curve & X-axis Between two Ordinates, Area of the Region Bounded by a Curve & Y-axis Between two Abscissa

1 MCQs with One Correct Answer

1. The area of the region [Adv. 2021]

$$\left\{(x,y): 0 \le x \le \frac{9}{4}, 0 \le y \le 1, x \ge 3y, x+y \ge 2\right\} \text{ is}$$

(a) $\frac{11}{32}$ (b) $\frac{35}{96}$ (c) $\frac{37}{96}$ (d) $\frac{13}{32}$

2. Let $g(x) = \cos x^2, f(x) = \sqrt{x}$, and $\alpha, \beta\ (\alpha < \beta)$ be the roots of the quadratic equation $18x^2 - 9\pi x + \pi^2 = 0$. Then the area (in sq. units) bounded by the curve $y = (gof)(x)$ and the lines $x = \alpha, x = \beta$ and $y = 0$, is: [Adv. 2018]

(a) $\frac{1}{2}(\sqrt{3}+1)$ (b) $\frac{1}{2}(\sqrt{3}-\sqrt{2})$

(c) $\frac{1}{2}(\sqrt{2}-1)$ (d) $\frac{1}{2}(\sqrt{3}-1)$

3. Let $f: [-1, 2] \to [0, \infty)$ be a continuous function such that $f(x) = f(1-x)$ for all $x \in [-1, 2]$

Let $R_1 = \int_{-1}^{2} xf(x)dx$, and R_2 be the area of the region bounded by $y = f(x), x = -1, x = 2$, and the x-axis. Then [2011]

(a) $R_1 = 2R_2$ (b) $R_1 = 3R_2$

(c) $2R_1 = R_2$ (d) $3R_1 = R_2$

4. Let the straight line $x = b$ divide the area enclosed by $y = (1-x)^2, y = 0$, and $x = 0$ into two parts $R_1\ (0 \le x \le b)$ and $R_2\ (b \le x \le 1)$ such that $R_1 - R_2 = \frac{1}{4}$. Then b equals [2011]

(a) $\frac{3}{4}$ (b) $\frac{1}{2}$ (c) $\frac{1}{3}$ (d) $\frac{1}{4}$

6 MCQs with One or More than One Correct Answer

5. Let $f: [0,1] \to [0,1]$ be the function defined by $f(x) = \frac{x^3}{3} - x^2 + \frac{5}{9}x + \frac{17}{36}$. Consider the square region $S = [0,1] \times [0,1]$. Let $G = \{(x, y) \in S : y > f(x)\}$ be called the green region and $R = \{(x, y) \in S : y < f(x)\}$ be called the red region. Let $L_h = \{(x, h) \in S : x \in [0,1]\}$ be the horizontal line drawn at a height $h \in [0,1]$. Then which of the following statements is(are) true? [Adv. 2023]

(a) There exists an $h \in \left[\frac{1}{4}, \frac{2}{3}\right]$ such that the area of the green region above the line L_h equals the area of the green region below the line L_h

(b) There exists an $h \in \left[\frac{1}{4}, \frac{2}{3}\right]$ such that the area of the red region above the line L_h equals the area of the red region below the line L_h

(c) There exists an $h \in \left[\frac{1}{4}, \frac{2}{3}\right]$ such that the area of the green region above the line L_h equals the area of the red region below the line L_h

(d) There exists an $h \in \left[\frac{1}{4}, \frac{2}{3}\right]$ such that the area of the red region above the line L_h equals the area of the green region below the line L_h

6. Let S be the area of the region enclosed by $y = e^{-x^2}$, $y = 0, x = 0$ and $x = 1$; then [2012]

(a) $S \ge \frac{1}{e}$ (b) $S \ge 1 - \frac{1}{e}$

(c) $S \le \frac{1}{4}\left(1 + \frac{1}{\sqrt{e}}\right)$ (d) $S \le \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right)$

7. Area of the region bounded by the curve $y = e^x$ and lines $x = 0$ and $y = e$ is [2009]

(a) $e-1$
(b) $\int_1^e \ln(e+1-y)\,dy$
(c) $e-\int_0^1 e^x dx$
(d) $\int_1^e \ln y\,dy$

8 Comprehension/Passage Based Questions

Consider the functions defined implicitly by the equation $y^3 - 3y + x = 0$ on various intervals in the real line. If $x \in (-\infty, -2) \cup (2, \infty)$, the equation implicitly defines a unique real valued differentiable function $y = f(x)$. If $x \in (-2, 2)$, the equation implicitly defines a unique real valued differentiable function $y = g(x)$ satisfying $g(0) = 0$. [2008]

8. If $f(-10\sqrt{2}) = 2\sqrt{2}$, then $f''(-10\sqrt{2}) =$

(a) $\frac{4\sqrt{2}}{7^3 3^2}$
(b) $-\frac{4\sqrt{2}}{7^3 3^2}$
(c) $\frac{4\sqrt{2}}{7^3 3}$
(d) $-\frac{4\sqrt{2}}{7^3 3}$

9. The area of the region bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$, where $-\infty < a < b < -2$, is

(a) $\int_a^b \frac{x}{3\left((f(x))^2-1\right)}dx + bf(b) - af(a)$
(b) $-\int_a^b \frac{x}{3\left((f(x))^2-1\right)}dx + bf(b) - af(a)$
(c) $\int_a^b \frac{x}{3\left((f(x))^2-1\right)}dx - bf(b) + af(a)$
(d) $-\int_a^b \frac{x}{3\left((f(x))^2-1\right)}dx - bf(b) + af(a)$

10. $\int_{-1}^{1} g'(x)\,dx =$

(a) $2g(-1)$ (b) 0 (c) $-2g(1)$ (d) $2g(1)$

10 Subjective Problems

11. Let $b \neq 0$ and for $j = 0, 1, 2, \ldots, n$, let S_j be the area of the region bounded by the y–axis and the curve $xe^{ay} = \sin by$, $\frac{jr}{b} \le y \le \frac{(j+1)\pi}{b}$. Show that $S_0, S_1, S_2, \ldots, S_n$ are in geometric progression. Also, find their sum for $a = -1$ and $b = \pi$. [2001 - 5 Marks]

12. Let A_n be the area bounded by the curve $y = (\tan x)^n$ and the lines $x = 0$, $y = 0$ and $x = \frac{\pi}{4}$. Prove that for $n > 2$, $A_n + A_{n-2} = \frac{1}{n-1}$ and deduce $\frac{1}{2n+2} < A_n < \frac{1}{2n-2}$. [1996 - 3 Marks]

13. Consider a square with vertices at (1, 1), (– 1, 1), (– 1, – 1) and (1, – 1). Let S be the region consisting of all points inside the square which are nearer to the origin than to any edge. Sketch the region S and find its area. [1995 - 5 Marks]

14. Find the area of the region bounded by the curve $C: y = \tan x$, tangent drawn to C at $x = \frac{\pi}{4}$ and the x-axis. [1988 - 5 Marks]

15. Find the area bounded by the x-axis, part of the curve $y = \left(1+\frac{8}{x^2}\right)$ and the ordinates at $x = 2$ and $x = 4$. If the ordinate at $x = a$ divides the area into two equal parts, find a. [1983 - 3 Marks]

16. For any real t, $x = \frac{e^t + e^{-t}}{2}$, $y = \frac{e^t - e^{-t}}{2}$ is a point on the hyperbola $x^2 - y^2 = 1$. Show that the area bounded by this hyperbola and the lines joining its centre to the points corresponding to t_1 and $-t_1$ is t_1. [1982 - 3 Marks]

Topic-2: Different Cases of Area Bounded Between the Curves

1 MCQs with One Correct Answer

1. Let $S = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \ge 0, y \ge 0, y^2 \le 4x, y^2 \le 12 - 2x$ and $3y + \sqrt{8x} \le 5\sqrt{8}\}$. If the area of the region S is $\alpha\sqrt{2}$, then α is equal to [Adv. 2024]

(a) $\frac{17}{2}$
(b) $\frac{17}{3}$
(c) $\frac{17}{4}$
(d) $\frac{17}{5}$

2. Let the functions $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = e^{x-1} - e^{-|x-1|}$ and $g(x) = \frac{1}{2}(e^{x-1} + e^{1-x})$. Then the area of the region in the first quadrant bounded by the curves $y = f(x)$, $y = g(x)$ and $x = 0$ is [Adv. 2020]

(a) $(2-\sqrt{3}) + \frac{1}{2}(e - e^{-1})$
(b) $(2+\sqrt{3}) + \frac{1}{2}(e - e^{-1})$
(c) $(2-\sqrt{3}) + \frac{1}{2}(e + e^{-1})$
(d) $(2+\sqrt{3}) + \frac{1}{2}(e + e^{-1})$

3. The area of the region $\{(x,y : xy \le 8, 1 \le y \le x^2\}$ is [Adv. 2018]

(a) $8\log_e 2 - \frac{14}{3}$ (b) $16\log_e 2 - \frac{14}{3}$

(c) $8\log_e 2 - \frac{7}{3}$ (d) $16\log_e 2 - 6$

4. Area of the region $\{(x,y) \in \mathbb{R}^2 : y \ge \sqrt{|x+3|}, 5y \le x+9 \le 15\}$ is equal to [Adv. 2016]

(a) $\frac{1}{6}$ (b) $\frac{4}{3}$ (c) $\frac{3}{2}$ (d) $\frac{5}{3}$

5. The area of the region described by $A = \{(x,y) : x^2 + y^2 \le 1 \text{ and } y^2 \le 1 - x\}$ is: [Adv. 2014]

(a) $\frac{\pi}{2} - \frac{2}{3}$ (b) $\frac{\pi}{2} + \frac{2}{3}$

(c) $\frac{\pi}{2} + \frac{4}{3}$ (d) $\frac{\pi}{2} - \frac{4}{3}$

6. The area enclosed by the curves $y = \sin x + \cos x$ and $y = |\cos x - \sin x|$ over the interval $\left[0, \frac{\pi}{2}\right]$ is [Adv. 2013]

(a) $4(\sqrt{2} - 1)$ (b) $2\sqrt{2}(\sqrt{2} - 1)$

(c) $2(\sqrt{2} + 1)$ (d) $2\sqrt{2}(\sqrt{2} + 1)$

7. Let f be a non-negative function defined on the interval $[0, 1]$. If $\int_0^x \sqrt{1 - (f'(t))^2}\, dt = \int_0^x f(t)\, dt, \quad 0 \le x \le 1,$ and $f(0) = 0$, then [2009]

(a) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

(b) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) > \frac{1}{3}$

(c) $f\left(\frac{1}{2}\right) < \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

(d) $f\left(\frac{1}{2}\right) > \frac{1}{2}$ and $f\left(\frac{1}{3}\right) < \frac{1}{3}$

8. The area of the region between the curves $y = \sqrt{\frac{1+\sin x}{\cos x}}$ and $y = \sqrt{\frac{1-\sin x}{\cos x}}$ bounded by the lines $x = 0$ and $x = \frac{\pi}{4}$ is [2008]

(a) $\int_0^{\sqrt{2}-1} \frac{t}{(1+t^2)\sqrt{1-t^2}}\, dt$

(b) $\int_0^{\sqrt{2}-1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}\, dt$

(c) $\int_0^{\sqrt{2}+1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}\, dt$

(d) $\int_0^{\sqrt{2}+1} \frac{t}{(1+t^2)\sqrt{1-t^2}}\, dt$

9. The area bounded by the parabolas $y = (x+1)^2$ and $y = (x-1)^2$ and the line $y = 1/4$ is [2005S]

(a) 4 sq. units (b) 1/6 sq. units

(c) 4/3 sq. units (d) 1/3 sq. units

10. The area enclosed between the curves $y = ax^2$ and $x = ay^2$ $(a > 0)$ is 1 sq. unit, then the value of a is [2004S]

(a) $1/\sqrt{3}$ (b) 1/2

(c) 1 (d) 1/3

11. The area bounded by the curves $y = \sqrt{x}$, $2y + 3 = x$ and x-axis in the 1st quadrant is [2003S]

(a) 9 (b) 27/4

(c) 36 (d) 18

12. The area bounded by the curves $y = |x| - 1$ and $y = -|x| + 1$ is [2002S]

(a) 1 (b) 2

(c) $2\sqrt{2}$ (d) 4

2 Integer Value Answer/Non-Negative Integer

13. Let $n \ge 2$ be a natural number and $f : [0, 1] \to \mathbb{R}$ be the function defined by [Adv. 2023]

$$f(x) = \begin{cases} n(1-2nx) & \text{if } 0 \le x \le \frac{1}{2n} \\ 2n(2nx-1) & \text{if } \frac{1}{2n} \le x \le \frac{3}{4n} \\ 4n(1-nx) & \text{if } \frac{3}{4n} \le x \le \frac{1}{n} \\ \frac{n}{n-1}(nx-1) & \text{if } \frac{1}{n} \le x \le 1 \end{cases}$$

If n is such that the area of the region bounded by the curves $x = 0$, $x = 1$, $y = 0$ and $y = f(x)$ is 4, then the maximum value of the function f is

14. Consider the functions $f, g : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x^2 + \frac{5}{12} \text{ and } g(x) = \begin{cases} 2\left(1 - \frac{4|x|}{3}\right), & |x| \le \frac{3}{4}, \\ 0, & |x| > \frac{3}{4}. \end{cases}$$

If α is the area of the region

$$\left\{(x, y) \in \mathbb{R} \times \mathbb{R} : |x| \le \frac{3}{4}, 0 \le y \le \min\{f(x), g(x)\}\right\},$$

then the value of 9α is ________. **[Adv. 2022]**

15. A farmer F_1 has a land in the shape of a triangle with vertices at P(0, 0), Q(1, 1) and R(2, 0). From this land, a neighbouring farmer F_2 takes away the region which lies between the side PQ and a curve of the form $y = x^n (n > 1)$. If the area of the region taken away by the farmer F_2 is exactly 30% of the area of ΔPQR, then the value of n is ________. **[Adv. 2018]**

16. Let $F(x) = \int_x^{x^2+\frac{\pi}{6}} 2\cos^2 t(dt)$ for all $x \in R$ and $f : \left[0, \frac{1}{2}\right] \to [0, \infty)$ be a continuous function. For $a \in \left[0, \frac{1}{2}\right]$, if $F'(a) + 2$ is the area of the region bounded by $x = 0, y = 0, y = f(x)$ and $x = a$, then $f(0)$ is **[Adv. 2015]**

3 Numeric/ New Stem Based Questions

Question Stem for Question Nos. 17 and 18

Consider the region $R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \ge 0 \text{ and } y^2 \le 4 - x\}$. Let F be the family of all circles that are contained in R and have centers on the x-axis. Let C be the circle that has largest radius among the circles in F. Let (α, β) be a point where the circle C meets the curve $y^2 = 4 - x$.

17. The radius of the circle C is ______. **[Adv. 2016]**

18. The value of α is ______ **[Adv. 2016]**

6 MCQs with One or More than One Correct Answer

19. If the line $x = \alpha$ divides the area of region $R = \left\{(x, y) \in R^2 : x^3 \le y \le x, 0 \le x \le 1\right\}$ into two equal parts, then **[Adv. 2017]**

(a) $0 < \alpha \le \frac{1}{2}$ (b) $\frac{1}{2} < \alpha < 1$

(c) $2\alpha^4 - 4\alpha^2 + 1 = 0$ (d) $\alpha^4 + 4\alpha^2 - 1 = 0$

20. For which of the following values of m, is the area of the region bounded by the curve $y = x - x^2$ and the line $y = mx$ equals 9/2? **[1999 - 3 Marks]**

(a) -4 (b) -2 (c) 2 (d) 4

7 Match the Following

21. Match the integrals in Column I with the values in Column II and indicate your answer by darkening the appropriate bubbles in the 4×4 matrix given in the *ORS*. **[2007 - 6 marks]**

Column I	Column II
(A) $\int_{-1}^{1} \frac{dx}{1+x^2}$	(p) $\frac{1}{2}\log\left(\frac{2}{3}\right)$
(B) $\int_{0}^{1} \frac{dx}{\sqrt{1-x^2}}$	(q) $2\log\left(\frac{2}{3}\right)$
(C) $\int_{2}^{3} \frac{dx}{1-x^2}$	(r) $\frac{\pi}{3}$
(D) $\int_{1}^{2} \frac{dx}{x\sqrt{x^2-1}}$	(s) $\frac{\pi}{2}$

8 Comprehension/Passage Based Questions

Consider the polynomial

$f(x) = 1 + 2x + 3x^2 + 4x^3$.

Let s be the sum of all distinct real roots of $f(x)$ and let $t = |s|$. **[2010]**

22. The real numbers lies in the interval

(a) $\left(-\frac{1}{4}, 0\right)$ (b) $\left(-11, -\frac{3}{4}\right)$

(c) $\left(-\frac{3}{4}, -\frac{1}{2}\right)$ (d) $\left(0, \frac{1}{4}\right)$

23. The area bounded by the curve $y = f(x)$ and the lines $x = 0$, $y = 0$ and $x = t$, lies in the interval

(a) $\left(\frac{3}{4}, 3\right)$ (b) $\left(\frac{21}{64}, \frac{11}{16}\right)$ (c) (9, 10) (d) $\left(0, \frac{21}{64}\right)$

24. The function $f'(x)$ is

(a) increasing in $\left(-t, -\frac{1}{4}\right)$ and decreasing in $\left(-\frac{1}{4}, t\right)$

(b) decreasing in $\left(-t, -\frac{1}{4}\right)$ and increasing in $\left(-\frac{1}{4}, t\right)$

(c) increasing in $(-t, t)$

(d) decreasing in $(-t, t)$

10 Subjective Problems

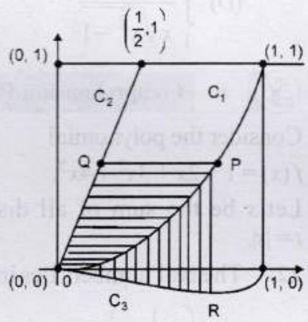
25. If $\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2+3a \\ 3b^2+3b \\ 3c^2+3c \end{bmatrix}$, $f(x)$ is a quadratic function and its maximum value occurs at a point V. A is a point of intersection of $y = f(x)$ with x-axis and point B is such that chord AB subtends a right angle at V. Find the area enclosed by $f(x)$ and chord AB. **[2005 - 6 Marks]**

26. Find the area bounded by the curves $x^2 = y$, $x^2 = -y$ and $y^2 = 4x - 3$. **[2005 - 4 Marks]**

27. Find the area of the region bounded by the curves $y = x^2$, $y = |2 - x^2|$ and $y = 2$, which lies to the right of the line $x = 1$. **[2002 - 5 Marks]**

28. Let $f(x)$ be a continuous function given by
$$f(x) = \begin{cases} 2x, & |x| \le 1 \\ x^2 + ax + b, & |x| > 1 \end{cases}$$ **[1999 - 10 Marks]**
Find the area of the region in the third quadrant bounded by the curves $x = -2y^2$ and $y = f(x)$ lying on the left of the line $8x + 1 = 0$.

29. Let C_1 and C_2 be the graphs of the functions $y = x^2$ and $y = 2x$, $0 \le x \le 1$ respectively. Let C_3 be the graph of a function $y = f(x)$, $0 \le x \le 1$, $f(0) = 0$. For a point P on C_1, let the lines through P, parallel to the axes, meet C_2 and C_3 at Q and R respectively (see figure.)



If for every position of P (on C_1), the areas of the shaded regions OPQ and ORP are equal, determine the function $f(x)$. **[1998 - 8 Marks]**

30. Let $f(x) = \text{Maximum } \{x^2, (1-x)^2, 2x(1-x)\}$, where $0 \le x \le 1$. Determine the area of the region bounded by the curves $y = f(x)$, x-axis, $x = 0$ and $x = 1$. **[1997 - 5 Marks]**

31. A curve $y = f(x)$ passes through the point $P(1,1)$. The normal to the curve at P is $a(y-1) + (x-1) = 0$. If the slope of the tangent at any point on the curve is proportional to the ordinate of the point, determine the equation of the curve. Also obtain the area bounded by the y-axis, the curve and the normal to the curve at P. **[1996 - 5 Marks]**

32. In what ratio does the x-axis divide the area of the region bounded by the parabolas $y = 4x - x^2$ and $y = x^2 - x$? **[1994 - 5 Marks]**

33. Sketch the region bounded by the curves $y = x^2$ and $y = \dfrac{2}{1+x^2}$. Find the area. **[1992 - 4 Marks]**

34. Sketch the curves and identify the region bounded by $x = \dfrac{1}{2}$, $x = 2$, $y = \ln x$ and $y = 2^x$. Find the area of this region. **[1991 - 4 Marks]**

35. Find all maxima and minima of the function
$y = x(x-1)^2, 0 \le x \le 2$ **[1989 - 5 Marks]**
Also determine the area bounded by the curve $y = x(x-1)^2$, the y-axis and the line $y = 2$.

36. Find the area bounded by the curves, $x^2 + y^2 = 25$, $4y = |4 - x^2|$ and $x = 0$ above the x-axis. **[1987 - 6 Marks]**

37. Sketch the region bounded by the curves $y = \sqrt{5 - x^2}$ and $y = |x - 1|$ and find its area. **[1985 - 5 Marks]**

38. Find the area of the region bounded by the x-axis and the curves defined by **[1984 - 4 Marks]**
$$y = \tan x, \ -\frac{\pi}{3} \le x \le \frac{\pi}{3}; \ y = \cot x, \ \frac{\pi}{6} \le x \le \frac{3\pi}{2}$$

Answer Key

Topic-1 : Curve & X-axis Between two Ordinates, Area of the Region Bounded by a Curve & Y-axis Between two Abscissa

1. (a) **2.** (d) **3.** (c) **4.** (b) **5.** (b, c, d) **6.** (a, b, d) **7.** (b, c, d) **8.** (b)
9. (a) **10.** (d)

Topic-2 : Different Cases of Area Bounded Between the Curves

1. (b) **2.** (a) **3.** (b) **4.** (c) **5.** (c) **6.** (b) **7.** (c) **8.** (b) **9.** (d) **10.** (a)
11. (a) **12.** (b) **13.** (8) **14.** (6) **15.** (4) **16.** (3) **17.** (1.5) **18.** (2.00) **19.** (b, c)
20. (b, d) **21.** (A) – s; (B) – s; (C) – p; (D) – r **22.** (c) **23.** (a) **24.** (b)

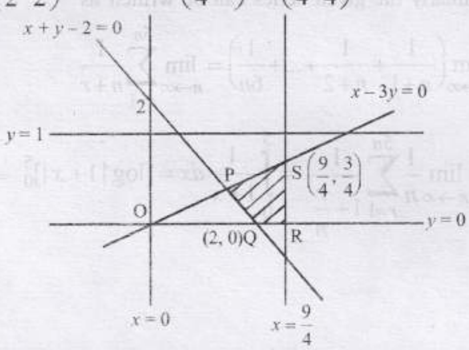
Hints & Solutions

Topic-1: Curve & X-axis Between two Ordinates, Area of the Region Bounded by a Curve & Y-axis Between two Abscissa

1. **(a)** Let $x+y-2=0, x=3y, y=1$ and $x=\frac{9}{4}$.

On solving, we get

$P\left(\frac{3}{2},\frac{1}{2}\right); Q(2,0); R\left(\frac{9}{4},0\right); S\left(\frac{9}{4},\frac{3}{4}\right)$



$$\text{Area} = \frac{1}{3}\int_{3/2}^{9/4} x\,dx - \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = \frac{1}{6}\left[x^2\right]_{3/2}^{9/4} - \frac{1}{8}$$

$$= \frac{1}{6}\times\frac{45}{16} - \frac{1}{8} = \frac{11}{32}$$

2. **(d)** Here, $18x^2 - 9\pi x + \pi^2 = 0$

$\Rightarrow (3x-\pi)(6x-\pi)=0$

$\Rightarrow \alpha = \frac{\pi}{6}, \beta = \frac{\pi}{3}$

Also, $gof(x) = \cos x$

$\therefore$ Req. area $= \int_{\pi/6}^{\pi/3} \cos x dx = \frac{\sqrt{3}-1}{2}$

3. **(c)** $R_1 = \int_{-1}^{2} x f(x)dx = \int_{-1}^{2}(1-x)f(1-x)dx$

$\left[\because \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$

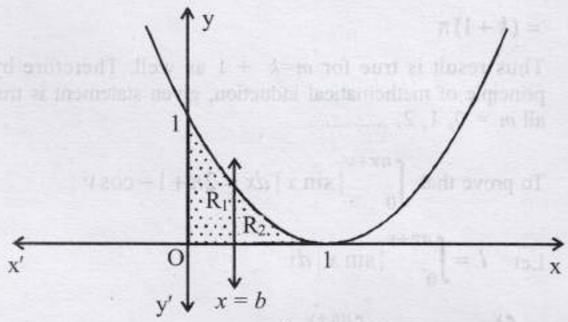
$\Rightarrow R_1 = \int_{-1}^{2}(1-x)f(x)dx$ [$\because f(x)=f(1-x)$ on $[-1,2]$]

Now, $R_1 + R_1 = \int_{-1}^{2} x f(x)dx + \int_{-1}^{2}(1-x)f(x)dx$

$\Rightarrow 2R_1 = \int_{-1}^{2} f(x)dx = R_2$

4. **(b)** $R_1 = \int_0^b (x-1)^2 dx = \left[\frac{(x-1)^3}{3}\right]_0^b = \frac{(b-1)^3+1}{3}$

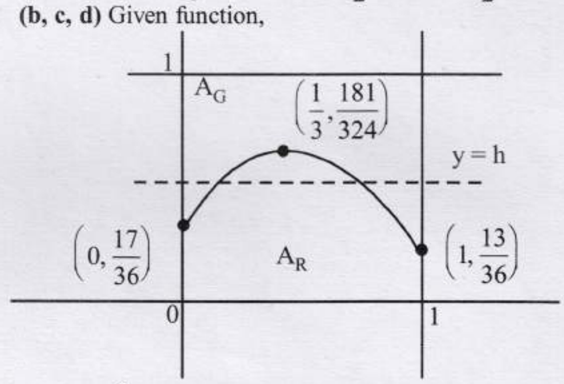
$R_2 = \int_b^1 (x-1)^2 dx = \left[\frac{(x-1)^3}{3}\right]_b^1 = -\frac{(b-1)^3}{3}$



$\because R_1 - R_2 = \frac{1}{4} \Rightarrow \frac{2(b-1)^3}{3} + \frac{1}{3} = \frac{1}{4}$

$\Rightarrow (b-1)^3 = -\frac{1}{8} \Rightarrow b-1 = \frac{-1}{2} \therefore b = \frac{1}{2}$

5. **(b, c, d)** Given function,



$f(x) = \frac{x^3}{3} - x^2 + \frac{5x}{9} + \frac{17}{36}$

$f'(x) = x^2 - 2x + \frac{5}{9} = 0 \Rightarrow x = \frac{1}{3}$

$f''(x) = 2x - 2 = \frac{2}{3} - 2 < 0$ at $x = \frac{1}{3}$

$\therefore f(x)$ is maximum at $x = \frac{1}{3}$

Now, $A_R = \int_0^1 f(x)\,dx = \int_0^1\left(\frac{x^3}{3} - x^2 + \frac{5}{5}x + \frac{17}{36}\right)dx = \frac{1}{2}$

$\therefore A_G = 1 - \frac{1}{2} = \frac{1}{2}$

(a) Since the area of the green region above the line L_h equals the area of the green region below the line L_h.

$\Rightarrow 1-h = h - \frac{1}{2} \Rightarrow h = \frac{3}{4}, \frac{3}{4} > \frac{2}{3}$

So, option (a) is incorrect

(b) Since, the area of the red region above the line L_h equals the area of the red region below the line L_h.

$\Rightarrow h = \frac{1}{2} - h \Rightarrow h = \frac{1}{4}$

So, option (b) is correct

(c) Since, the area of the green region above the line Lh equals the area of the red region below the line Lh.

When $h = \frac{181}{324}, A_R = \frac{1}{2}, A_G < \frac{1}{2}$

$h = \frac{13}{36}, A_R = \frac{1}{2}, A_G < \frac{1}{2}$

$A_R = A_G$ for some $\left(\frac{13}{36}, \frac{181}{324}\right)$

So, option (c) is correct

(d) $\because$ option (c) is correct $\Rightarrow$ option (d) is also correct

6. **(a, b, d)** The given curve $y = e^{-x^2}$

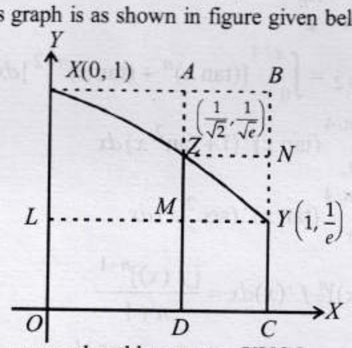
Draw a rough sketch of curve

at $x = 0, y = 1$ and at $x = 1, y = 1/e$ $\because y = e^{-x^2}$

$\Rightarrow \frac{dy}{dx} = -2xe^{-x^2} < 0 \ \forall \ x \in (0,1)$

$\therefore y = e^{-x^2}$ is decreasing on (0, 1)

Hence its graph is as shown in figure given below



Now, S = area exclosed by curve = XYCO

and area of rectangle OCYL = $\frac{1}{e}$

Clearly $S > \frac{1}{e}$ $\therefore$ A is true.

For $x \in [0,1] \Rightarrow x^2 < x$

$\Rightarrow -x^2 > -x \Rightarrow e^{-x^2} \geq e^{-x} \ \forall \ x \in [0,1]$

$\Rightarrow \int_0^1 e^{-x^2} dx > \int_0^1 e^{-x} dx = 1 - \frac{1}{e} \Rightarrow S > 1 - \frac{1}{e}$ $\therefore$ (b) is true.

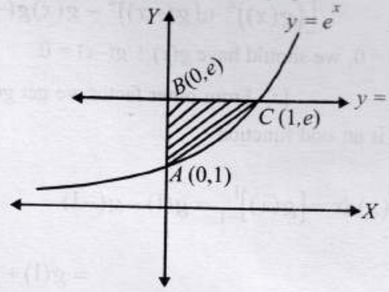
Now S < area of rectangle XADO + area of rectangle ZDCN

$\Rightarrow S < \frac{1}{\sqrt{2}} \times 1 + \left(1 - \frac{1}{\sqrt{2}}\right)\frac{1}{\sqrt{e}}$

$\therefore S < \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right)$ $\because$ (d) is true

Also as $\frac{1}{4}\left(1 + \frac{1}{\sqrt{e}}\right) < 1 - \frac{1}{e}$ $\therefore$ (c) is incorrect.

7. **(b, c, d)** The area bounded by the curve $y = e^x$ and lines $x = 0$ and $y = e$ is as shown in the graph.



Required area $= \int_0^1 (e - e^x)dx = e - \int_0^1 e^x dx. = 1$

Also required area

$= \int_0^e x\, dy = \int_1^e \ln y\, dy$ (where $e^x = y \Rightarrow x = \ln y$)

$= \int_1^e \ln(e + 1 - y)dy \left[\because \int_a^b f(x)dx = \int_a^b f(a + b - x)dx\right]$

8. **(b)** We have $y^3 - 3y + x = 0 \Rightarrow 3y^2\frac{dy}{dx} - 3\frac{dy}{dx} + 1 = 0$

$\Rightarrow \frac{dy}{dx} = \frac{1}{3(1 - y^2)}$ or $f'(x) = \frac{1}{3\left[1 - [f(x)]^2\right]}$

Also $3y^2\frac{d^2y}{dx^2} + 6y\left(\frac{dy}{dx}\right)^2 - 3\frac{d^2y}{dx^2} = 0$

$\Rightarrow \frac{d^2y}{dx^2} = \frac{2y}{1 - y^2}\left(\frac{dy}{dx}\right)^2 \Rightarrow f''(x) = \frac{2f(x)}{9\left[1 - [f(x)]^2\right]^3}$

$\therefore f''(-10\sqrt{2}) = \frac{-4\sqrt{2}}{3^2 \times 7^3}$

9. **(a)** For $x < -2$

we have, $3y - y^3 < -2 \Rightarrow y^3 - 3y - 2 > 0$

$\Rightarrow (y + 1)^2 (y - 2) > 0 \Rightarrow y > 2 \ \forall \ x < -2$

$\Rightarrow f(x)$ is positive $\forall \ x < -2$

Hence required area $= \int_a^b f(x)dx = \int_a^b 1.f(x)dx$

$= x\, f(x)]_a^b - \int_a^b x\, f'(x)dx$

$= b\, f(b) - a\, f(a) - \int_a^b \frac{x.1}{3\left[1 - (f(x))^2\right]}dx$

$= \int_a^b \frac{x}{3\left[(f(x))^2 - 1\right]}dx + b\, f(b) - a\, f(a)$

10. **(d)** For $y = g(x)$, we have $y^3 - 3y + x = 0$

$\Rightarrow [g(x)]^3 - 3[g(x)] + x = 0$ ...(i)

Putting $x = -x$, we get

$\Rightarrow [g(-x)]^3 - 3[g(-x)] - x = 0$ ...(ii)

Adding equations (i) and (ii) we get

$[g(x)]^3 + [g(-x)]^3 - 3\{[g(x)] + [g(-x)]\} = 0 \frac{n!}{r!(n-r)!}$

$$\Rightarrow [g(x)+g(-x)]$$
$$\left[(g(x))^2+(g(-x))^2-g(x)g(-x)-3\right]=0$$

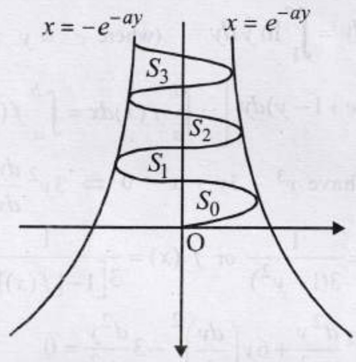
For $g(0) = 0$, we should have $g(x) + g(-x) = 0$

[$\because$ From other factor we get $g(0) = \pm\sqrt{3}$]

$\Rightarrow g(x)$ is an odd function

$$\therefore \int_{-1}^{1} g'(x)dx = [g(x)]_{-1}^{1} = g(1)-g(-1)$$
$$= g(1)+g(1) = 2\,g(1).$$

11. It is given that $x = \sin by.\, e^{-ay} \Rightarrow -e^{-ay} \le x \le e^{-ay}$
The figure is drawn taking a and b both +ve. The given curve oscillates between $x = e^{-ay}$ and $x = -e^{-ay}$



Clearly, $S_j = \int_{\frac{j\pi}{b}}^{\frac{(j+1)\pi}{b}} \sin by.e^{-ay}.dy$

Integrating by parts, $I = \int \sin by.e^{-ay}.dy$

We get $I = -\dfrac{e^{-ay}}{a^2+b^2}(a\sin by + b\cos by)$

$$\text{So, } S_j = \left| -\frac{1}{a^2+b^2}\left[e^{-a\frac{(j+1)\pi}{b}}\{a\sin(j+1)\pi + b\cos(j+1)\pi - e^{\frac{-aj\pi}{b}}(a\sin j\pi + b\cos j\pi)\right]\right|$$

$$\Rightarrow S_j = \left| -\frac{1}{a^2+b^2}\left[e^{-\frac{a}{b}(j+1)\pi} b(-1)^{j+1} - e^{-\frac{a}{b}j\pi} b(-1)^j \right]\right|$$

$$= \left| \frac{b.(-1)^j e^{-\frac{a}{b}j\pi}}{a^2+b^2}\left(e^{-\frac{a}{b}\pi}+1\right)\right| = b.\frac{e^{-\frac{a}{b}j\pi}}{a^2+b^2}\left(e^{-\frac{a}{b}\pi}+1\right)$$

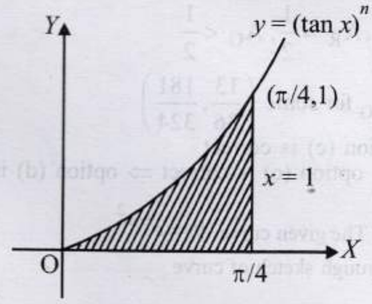
Now, $\dfrac{S_j}{S_{j-1}} = \dfrac{e^{-\frac{a}{b}j\pi}}{e^{-\frac{a}{b}(j-1)\pi}} = e^{-\frac{a}{b}\pi}$ = constant

$\Rightarrow S_o, S_1, S_2, \ldots\ldots\ldots S_j$ form a G.P.

For $a = -1$ and $b = \pi$ $\quad S_j = \dfrac{\pi e^j}{(1+\pi^2)}(1+e)$

$$\Rightarrow \sum_{j=0}^{n} S_j = \frac{\pi(1+e)}{(1+\pi^2)}.\frac{(e^{(n+1)}-1)}{(e-1)}.$$

12. We have $A_n = \int_0^{\pi/4} (\tan x)^n dx$



Since $0 < \tan x < 1$, when $0 < x < \pi/4$, we have

$0 < (\tan x)^{n+1} < (\tan x)^n$ for each $n \in N$

$$\Rightarrow \int_0^{\pi/4} (\tan x)^{n+1} dx < \int_0^{\pi/4} (\tan x)^n dx$$

$\Rightarrow A_{n+1} < A_n$

Now, for $n > 2$

$$A_n + A_{n+2} = \int_0^{\pi/4} [(\tan x)^n + (\tan x)^{n+2}]dx$$
$$= \int_0^{\pi/4} (\tan x)^n (1+\tan^2 x)dx$$
$$= \int_0^{\pi/4} (\tan x)^n (\sec^2 x)dx$$

$$\because \int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1}$$

$$A_n + A_{n+2} = \left[\frac{1}{(n+1)}(\tan x)^{n+1}\right]_0^{\pi/4} = \frac{1}{(n+1)}(1-0)$$

Since $A_{n+2} < A_{n+1} < A_n$, we get, $A_n + A_{n+2} < 2A_n$

$$\Rightarrow \frac{1}{n+1} < 2A_n \Rightarrow \frac{1}{2n+2} < A_n \quad \text{........(i)}$$

Also for $n > 2, A_n + A_n < A_n + A_{n-2} = \dfrac{1}{n-1}$

$$\Rightarrow 2A_n < \frac{1}{n-1}$$

$$\Rightarrow A_n < \frac{1}{2n-2} \quad \text{........(ii)}$$

Combining (i) and (ii) we get

$$\frac{1}{2n+2} < A_n < \frac{1}{2n-2}$$ Hence Proved.

13. Let us consider any point $P\ (x, y)$ inside the square such that its distance from origin $\le$ its distance from any of the edges say AD

$\therefore OP \le PM$ or $\sqrt{(x^2+y^2)} < 1-x$

or $\quad y^2 \le -2\left(x-\frac{1}{2}\right)$ .... (i)

Above represents all points within and on the parabola 1. If we consider the edges BC then $OP < PN$ will imply

$y^2 \le 2\left(x+\frac{1}{2}\right)$ .... (ii)

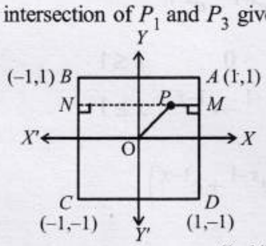
Similarly if we consider the edges AB and CD, we will have

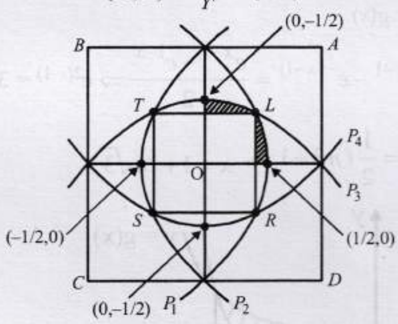
$x^2 \le -2\left(y-\frac{1}{2}\right)$ .... (iii)

$x^2 \le 2\left(y+\frac{1}{2}\right)$ .... (iv)

Hence S consists of the region bounded by four parabolas meeting the axes at $\left(\pm\frac{1}{2}, 0\right)$ and $\left(0, \pm\frac{1}{2}\right)$

The point L is intersection of P_1 and P_3 given by (i) and (iii).





$y^2 - x^2 = -2(x-y) = 2(y-x) \;\therefore\; y-x=0 \;\therefore\; y=x$

$\therefore \quad x^2+2x-1=0 \Rightarrow (x+1)^2 = 2$

$\therefore \quad x=\sqrt{2}-1$ as x is +ve $\therefore \quad L$ is $(\sqrt{2}-1, \sqrt{2}-1)$

$\therefore$ Total area $= 4\left[\text{square of side } (\sqrt{2}-1) + 2\int_{\sqrt{2}-1}^{1/2} y\,dx\right]$

$= 4\left\{(\sqrt{2}-1)^2 + 2\int_{\sqrt{2}-1}^{1/2}\sqrt{(1-2x)}\,dx\right\}$

$= 4\left[3-2\sqrt{2} - \frac{2}{2}\cdot\frac{2}{3}\{(1-2x)^{3/2}\}_{\sqrt{2}-1}^{1/2}\right]$

$= 4(3-2\sqrt{2})\left[1+\frac{2}{3}\sqrt{(3-2\sqrt{2})}\right]$

$= \frac{4}{3}(3-2\sqrt{2})(1+2\sqrt{2}) = \frac{4}{3}[(4\sqrt{2}-5)] = \frac{16\sqrt{2}-20}{3}$

14. The given curve is $y = \tan x$ ...(i)

Let A be the point on (i) where $x = \pi/4$

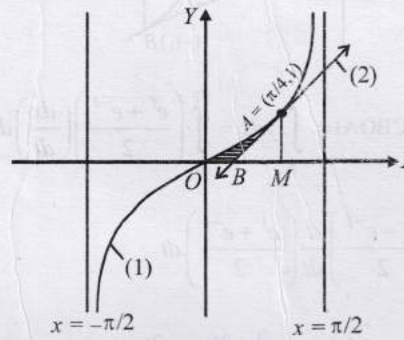
$\therefore \quad y = \tan \pi/4 = 1$

So, co-ordinates of A are $(\pi/4, 1)$

$\therefore$ Equation of tangent at A is $y-1 = 2(x-\pi/4)$

or $y = 2x+1-\pi/2$ ...(ii)

The graph of (1) and (2) are as shown in the figure.



Tangent (2) meets x-axis at, $L\left(\frac{\pi-2}{4}, 0\right)$

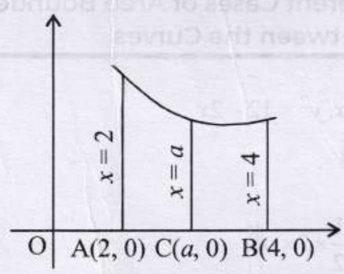
Now the required area = shaded area

$=$ Area $OAMO - Ar(\Delta ABM)$

$= \int_0^{\pi/4}\tan x\,dx - \frac{1}{2}(OM-OB)AM$

$= [\log\sec x]_0^{\pi/4} - \frac{1}{2}\left\{\frac{\pi}{4} - \frac{\pi-2}{4}\right\}.1 = \frac{1}{2}\left[\log 2 - \frac{1}{2}\right]$ sq.units.

15. The equation of curve is, $y = 1+\frac{8}{x^2}$



Req. area $= \int_2^4 y\,dx = \int_2^4\left(1+\frac{8}{x^2}\right)dx = \left[x-\frac{8}{x}\right]_2^4 = 4$

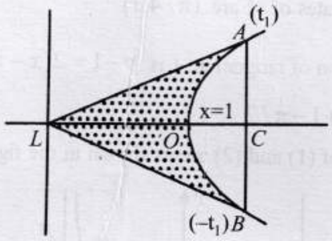
If $x = a$ bisects the area then we have

$\int_2^a\left(1+\frac{8}{x^2}\right)dx = \left[x-\frac{8}{x}\right]_2^a = \left[a-\frac{8}{a}-2+4\right] = \frac{4}{2}$

$$\Rightarrow\ a-\frac{8}{a}=0 \Rightarrow a^2=0\ \Rightarrow\ a=\pm 2\sqrt{2}$$

Since $2<a<4\ \therefore a=2\sqrt{2}$

16. Let $P(t_1)$ and $Q(-t_1)$ be two points on the hyperbola.



$$\text{Area (ACBOA)}=\int_{-t_1}^{t_1} y\,dx=\int_{-t_1}^{t_1}\left(\frac{e^t+e^{-t}}{2}\right)\left(\frac{dx}{dt}\right)dt$$

$$=\int_{-t_1}^{t_1}\left(\frac{e^t-e^{-t}}{2}\right)\frac{d}{dt}\left(\frac{e^t+e^{-t}}{2}\right).dt$$

$$=\int_{-t_1}^{t_1}\left(\frac{e^t-e^{-t}}{2}\right)dt=\int_{-t_1}^{t_1}\frac{e^{2t}+e^{-2t}-2}{4}dt$$

$$=\frac{2}{8}(e^{2t_1}-e^{-2t_1}-4t_1)=\frac{e^{2t_1}-e^{-2t_1}}{4}-t_1 \quad \ldots \text{(i)}$$

$$\text{Area of } \Delta LAC=\frac{1}{2}LC\times AB\ =LC\times AC$$

$$=\frac{e^{t_1}+e^{-t_1}}{2}\times\frac{e^{t_1}-e^{-t_1}}{2}=\frac{e^{2t_1}-e^{-2t_1}}{4} \quad \ldots \text{(ii)}$$

$\therefore$ The required area $=Ar(\Delta LAB)-Ar(ACBOA)$

$$=\frac{e^{2t_1}-e^{-2t_1}}{4}-\frac{e^{2t_1}-e^{-2t_1}}{4}+t_1=t_1$$

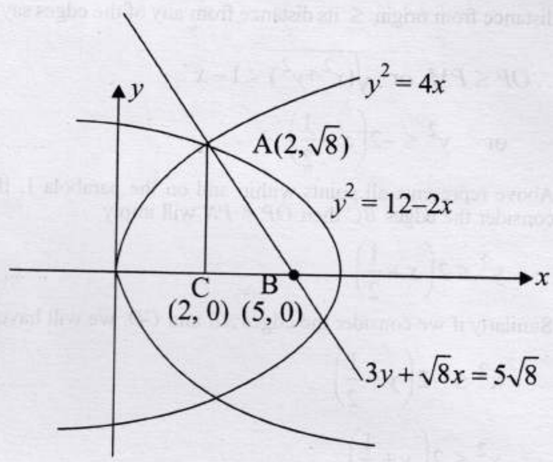
Topic-2: Different Cases of Area Bounded Between the Curves

1. (b) We have, $y^2=4x,\ y^2=12-2x$

$$\Rightarrow x=2,\ y=\sqrt{8}$$

$$A=\int_0^2 2\sqrt{x}dx+\frac{1}{2}\times 3\times\sqrt{8}$$

$$=\left[2\times\frac{2}{3}\times x^{\frac{3}{2}}\right]_0^2+3\sqrt{2}=\frac{4}{3}\times 2\sqrt{2}+3\sqrt{2}=\frac{17}{3}\sqrt{2}$$



$$\because A=\alpha\sqrt{2}\Rightarrow\alpha=\frac{17}{3}$$

2. (a) $\because f(x)=e^{x-1}-e^{|x-1|}$

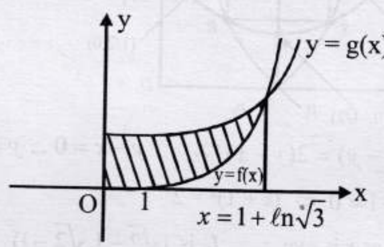
$$\therefore f(x)=\begin{cases}0 & x\le 1\\ e^{x-1}-e^{1-x} & x\ge 1\end{cases}$$

and $g(x)=\frac{1}{2}\left(e^{x-1}+e^{1-x}\right)$

If $f(x)=g(x)$

$$\Rightarrow\quad e^{x-1}-e^{-(x-1)}=\frac{e^{x-1}+e^{1-x}}{2}\Rightarrow e^{2(x-1)}=3$$

$$\Rightarrow\quad x=\frac{1}{2}\ell n3+1\Rightarrow x=1+\ell n\sqrt{3}$$



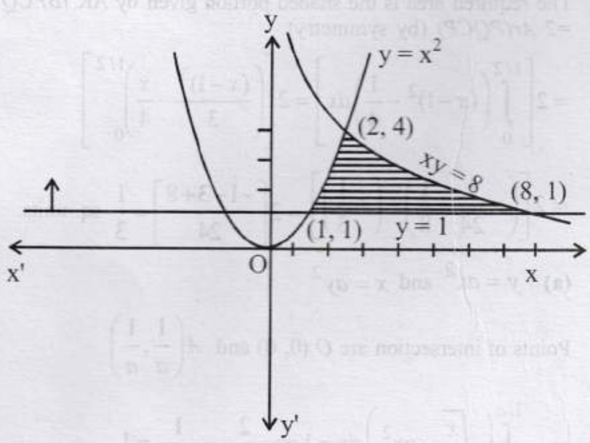
$$\text{So bounded area }=\int_0^{\frac{1}{2}\ell n3+1} g(x)dx-\int_1^{\frac{1}{2}\ell n3+1} f(x)dx$$

$$=\frac{1}{2}\left[e^{x-1}-e^{1-x}\right]_0^{\frac{1}{2}\ell n3+1}-\left[e^{x-1}+e^{1-x}\right]_1^{\frac{1}{2}\ell n3+1}$$

$$=2-\sqrt{3}+\frac{1}{2}\left(e-\frac{1}{e}\right)$$

3. (b) $xy\le 8,\quad 1\le y\le x^2$

Intersection points of $xy=8$ and $y=1$ is $(8, 1)$; $xy=8$ and $y=x^2$ is $(2, 4)$ and $y=x^2$ and $y=1$ is $(1, 1)$



Required area $= \int_1^2 x^2 dx + \int_2^8 \frac{8}{x} dx - \int_1^8 1 dx$

$= \left[\frac{x^3}{3}\right]_1^2 + [8\ln x]_2^8 - [x]_1^8 = \frac{8}{3} - \frac{1}{3} + 8\ln 8 - 8\ln 2 - (8-1)$

$= \frac{7}{3} + 24\ln 2 - 8\ln 2 - 7 = 16\ln 2 - \frac{14}{3}$

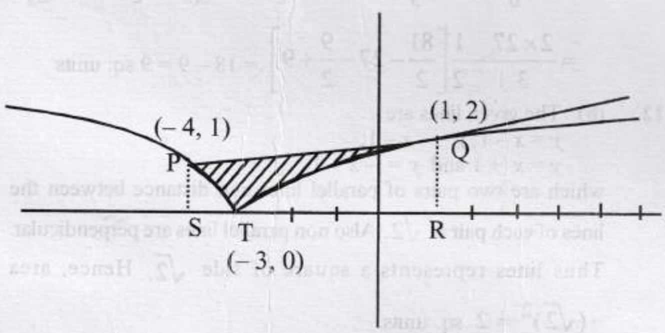
4. (c) $y \geq \sqrt{|x+3|} \Rightarrow y^2 = |x+3|$

$\Rightarrow y^2 = \begin{cases} -(x+3) \text{ if } x < -3 \\ (x+3) \text{ if } x \geq -3 \end{cases}$ ....(i)

Also $y \leq \frac{x+9}{5}$ and $x \leq 6$ ....(ii)

Solving (i) and (ii), we get intersection points as (1, 2), (6, 3), (–4, 1), (–39, –6)

The graph of given region is as follows-



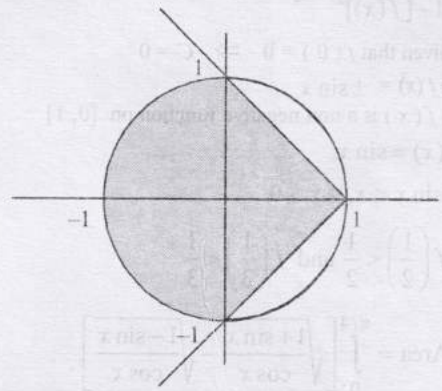
Required area = Area (trap PQRS) – Area (PST + TQR)

$= \frac{1}{2} \times (1+2) \times 5 - \left[\int_{-4}^{-3} \sqrt{-x-3}\, dx + \int_{-3}^{1} \sqrt{x+3}\, dx\right]$

$= \frac{15}{2} - \left[\left(\frac{2(-x-3)^{3/2}}{-3}\right)_{-4}^{-3} + \left(\frac{2(x+3)^{3/2}}{3}\right)_{-3}^{1}\right]$

$= \frac{15}{2} - \left[\frac{2}{3} + \frac{16}{3}\right] = \frac{15}{2} - 6 = \frac{3}{2}$ sq.units

5. (c) Given curves are $x^2 + y^2 = 1$ and $y^2 = 1 - x$.
Intersecting points are $x = 0, 1$



Area of shaded portion is the required area.
So, Required Area = Area of semi-circle
+ Area bounded by parabola

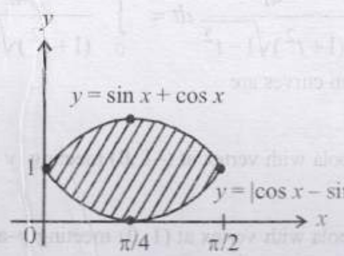
$= \frac{\pi r^2}{2} + 2\int_0^1 \sqrt{1-x}dx = \frac{\pi}{2} + 2\int_0^1 \sqrt{1-x}\, dx$

($\because$ radius of circle = 1)

$= \frac{\pi}{2} + 2\left[\frac{(1-x)^{3/2}}{-3/2}\right]_0^1 = \frac{\pi}{2} - \frac{4}{3}(-1) = \frac{\pi}{2} + \frac{4}{3}$ sq. unit

6. (b) The rough graph of $y = \sin x + \cos x$ and $y = |\cos x - \sin x|$ suggest the required area is

$= \int_0^{\pi/2} [(\sin x + \cos x) - |\cos x - \sin x|]\, dx$



$= \int_0^{\pi/4} 2\sin x\, dx + \int_{\pi/4}^{\pi/2} 2\cos x\, dx$

$= 2\left[(-\cos x)_0^{\pi/4} + (\sin x)_{\pi/4}^{\pi/2}\right] = 2\sqrt{2}(\sqrt{2}-1)$

7. (c) Given that f is a non negative function defined on [0, 1] and $\int_0^x \sqrt{1-(f'(t))^2}\, dt = \int_0^x f(t)dt,\ 0 \leq x \leq 1$

Differentiating both sides with respect to x, we get

$\sqrt{1-[f'(x)]^2} = f(x)$

$\Rightarrow 1 - [f'(x)]^2 = [f(x)]^2 \Rightarrow [f'(x)]^2 = 1 - [f(x)]^2$

$\Rightarrow \frac{d}{dx} f(x) = \pm\sqrt{1-[f(x)]^2} \Rightarrow \pm \frac{d\, f(x)}{\sqrt{1-[f(x)]^2}} = dx$

Integrating both sides with respect to x, we get

$$\pm\int \frac{d\,f(x)}{\sqrt{1-[f(x)]^2}}=\int dx \Rightarrow \pm\sin^{-1} f(x)=x+C$$

$\because$ Given that $f(0)=0 \Rightarrow C=0$

Hence $f(x)=\pm\sin x$

But as $f(x)$ is a non negative function on $[0, 1]$

$\therefore\ f(x)=\sin x.$

Now $\sin x < x, \forall\, x>0$

$$\therefore\quad f\left(\frac{1}{2}\right)<\frac{1}{2} \text{ and } f\left(\frac{1}{3}\right)<\frac{1}{3}.$$

8. **(b)** Area $=\displaystyle\int_0^{\pi/4}\left[\sqrt{\frac{1+\sin x}{\cos x}}-\sqrt{\frac{1-\sin x}{\cos x}}\right],$

$$\left[\because 1+\frac{\sin(x)}{\cos(x)}>\frac{1-\sin(x)}{\cos(x)}>0\right]$$

$$=\int_0^{\pi/4}\left(\sqrt{\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}}-\sqrt{\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}}\right)dx=\int_0^{\pi/4}\frac{2\tan\frac{x}{2}}{\sqrt{1-\tan^2\frac{x}{2}}}dx$$

Let $\tan\frac{x}{2}=t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}dx=dt \Rightarrow dx=\frac{2}{1+t^2}dt$

When $x=0$, then $t=0$ and when $x=\frac{\pi}{4}$, then $t=\tan\frac{\pi}{8}$

$$\therefore A=\int_0^{\tan\frac{\pi}{8}}\frac{4t}{(1+t^2)\sqrt{1-t^2}}dt=\int_0^{\sqrt{2}-1}\frac{4t}{(1+t^2)\sqrt{1-t^2}}dt$$

9. **(d)** The given curves are

$y=(x+1)^2$ ...(i)

upward parabola with vertex at $(-1, 0)$ meeting y–axis at $(0, 1)$

$y=(x-1)^2$ ...(ii)

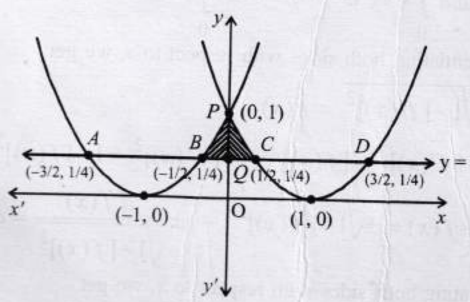
upward parabola with vertex at $(1, 0)$ meeting y–axis at $(0, 1)$

$y=1/4$ ... (iii)

a line parallel to x–axis meeting (i) at $\left(-\frac{1}{2},\frac{1}{4}\right),\left(-\frac{3}{2},\frac{1}{4}\right)$

and meeting (ii) at $\left(\frac{3}{2},\frac{1}{4}\right),\left(\frac{1}{2},\frac{1}{4}\right).$

The graph is as shown



The required area is the shaded portion given by AR $(BPCQB)$ $=2\,Ar(PQCP)$ (by symmetry)

$$=2\left[\int_0^{1/2}\left((x-1)^2-\frac{1}{4}\right)dx\right]=2\left[\left(\frac{(x-1)^3}{3}-\frac{x}{4}\right)_0^{1/2}\right]$$

$$=2\left[\left(-\frac{1}{24}-\frac{1}{8}\right)-\left(-\frac{1}{3}\right)\right]=2\left[\frac{-1-3+8}{24}\right]=\frac{1}{3}\text{ sq. units.}$$

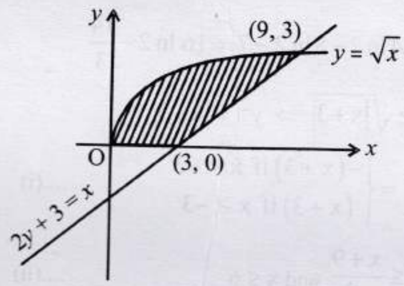
10. **(a)** $y=ax^2$ and $x=ay^2$

Points of intersection are $O\,(0, 0)$ and $A\left(\frac{1}{a},\frac{1}{a}\right)$

$$\Rightarrow \int_0^{1/a}\left(\sqrt{\frac{x}{a}}-ax^2\right)dx=1 \Rightarrow \frac{2}{3a^2}-\frac{1}{3a^2}=1$$

$$\Rightarrow \frac{1}{3a^2}=1,\ \therefore\ a=\pm\frac{1}{\sqrt{3}}$$

11. **(a)** The curves given are $y=\sqrt{x}$, $2y+3=x$, and x-axis i.e. $y=0$



$$\text{Area}=\int_0^9\sqrt{x}dx-\int_3^9\frac{x-3}{2}dx=\left[\frac{2x^{3/2}}{3}\right]_0^9-\frac{1}{2}\left[\frac{x^2}{2}-3x\right]_3^9$$

$$=\frac{2\times 27}{3}-\frac{1}{2}\left[\frac{81}{2}-27-\frac{9}{2}+9\right]=18-9=9\text{ sq. units}$$

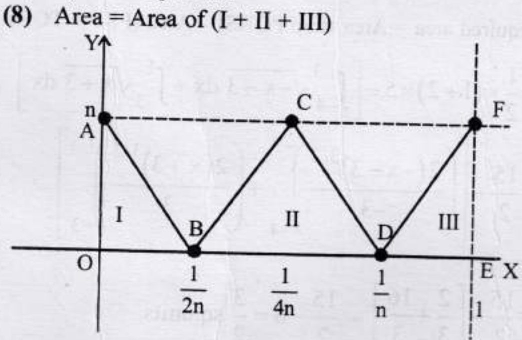
12. **(b)** The given lines are

$y=x-1;\ y=-x-1;$

$y=x+1$ and $y=-x+1$

which are two pairs of parallel lines and distance between the lines of each pair is $\sqrt{2}$. Also non parallel lines are perpendicular. Thus lines represents a square of side $\sqrt{2}$. Hence, area $=(\sqrt{2})^2=2$ sq. units.

13. **(8)** Area = Area of (I + II + III)

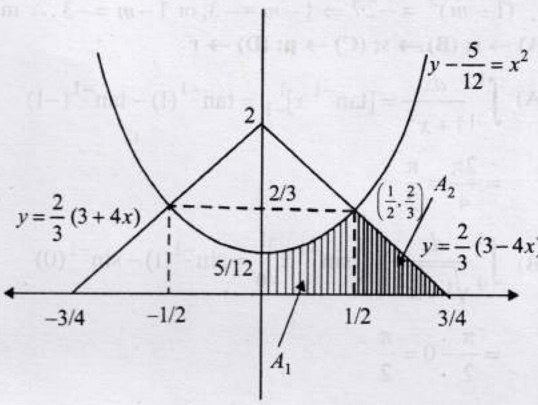


$$4=\frac{1}{2}\times\frac{1}{2n}\times n+\frac{1}{2}\times\frac{1}{2n}\times n+\frac{1}{2}\left(1-\frac{1}{n}\right)\times n$$

$$4=\frac{1}{4}+\frac{1}{4}+\frac{n-1}{2}\Rightarrow n=8$$

$\therefore$ Maximum value of f(x) = 8.

14. (6)



$$y=x^2+\frac{5}{12} \qquad \text{...(i)}$$

$$y=\frac{2}{3}(3-4x) \qquad \text{...(ii)}$$

Solving equations (i) and (ii), we get $x=\frac{1}{2}, y=\frac{2}{3}$

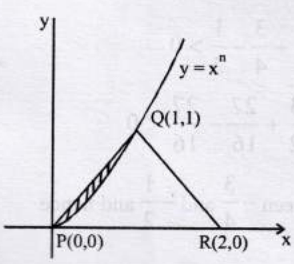
$\therefore$ Required area = 2 . (A_1+A_2)

$$\Rightarrow \quad \alpha=2\left(\int_0^{1/2}\left(x^2+\frac{5}{12}\right)dx+\frac{1}{2}\left(\frac{3}{4}-\frac{1}{2}\right).\frac{2}{3}\right)$$

$$\Rightarrow \quad \alpha=2\left(\left(\frac{x^3}{3}+\frac{5x}{12}\right)\Bigg|_0^{1/2}+\frac{1}{2}.\frac{1}{4}.\frac{2}{3}\right)$$

$$\Rightarrow \quad \alpha=2\left(\frac{1}{4}+\frac{1}{12}\right) \Rightarrow \alpha=\frac{2}{3} \Rightarrow 9\alpha=6$$

15. (4)



$$\text{Shaded area}=\frac{30}{100}\times Ar(\Delta PQR)$$

$$\Rightarrow \int_0^1(x-x^n)dx=\frac{3}{10}\times\frac{1}{2}\times 2\times 1$$

$$\Rightarrow \left(\frac{x^2}{2}-\frac{x^{n+1}}{n+1}\right)_0^1=\frac{3}{10}\Rightarrow\frac{1}{2}-\frac{1}{n+1}=\frac{3}{10} \Rightarrow n=4$$

16. (3) $F(x)=\int_x^{x^2+\pi/6}2\cos^2 t\,dt$

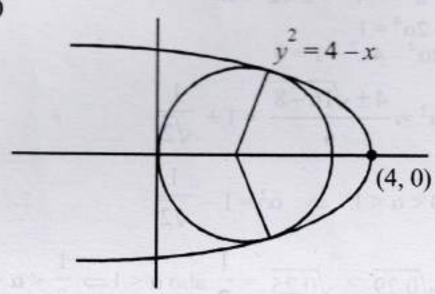
$$F'(\alpha)=2\cos^2\left(\alpha^2+\frac{\pi}{6}\right).2\alpha-2\cos^2\alpha$$

$$F'(\alpha)+2=\int_0^\alpha f(x)dx \Rightarrow F''(\alpha)=f(\alpha)$$

$$\therefore \quad f(\alpha)=4\alpha.2\cos\left(\alpha^2+\frac{\pi}{6}\right).\left[-\sin\left(\alpha^2+\frac{\pi}{6}\right)\right].2\alpha$$
$$+4\cos^2\left(\alpha^2+\frac{\pi}{6}\right)-4\cos\alpha\,(-\sin\alpha)$$

$$\therefore \quad f(0)=4\cos^2\frac{\pi}{6}=4\times\frac{3}{4}=3$$

17. (1.50)



Since C be the circle that has largest radius so, it touches the y-axis at (0,0) and centre at x-axis.

Let the equation of circle be

$x^2+y^2+\lambda x=0$

For point of intersection of circle & parabola $y^2=4-x$.

$x^2+4-x+\lambda x=0$

$\Rightarrow \quad x^2+x(\lambda-1)+4=0 \qquad \text{. . . (i)}$

For tangency : $\Delta=0$

$\Rightarrow \quad (\lambda-1)^2-16=0$

$\Rightarrow \quad \lambda=5$ (rejected) or $\lambda=-3$

$\therefore \quad$ Equation of circle : $x^2+y^2-3x=0$

$$\text{Radius}=\frac{3}{2}=1.5$$

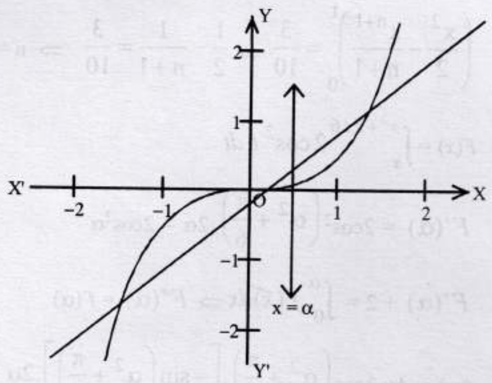
18. (2.00) For point of intersection.

Put $\lambda=-3$ in equation (i)

$x^2-4x+4=0$

$\Rightarrow \quad x=2$ so $\alpha=2$

19. (b, c) $\int_0^\alpha(x-x^3)dx=\frac{1}{2}\int_0^1(x-x^3)dx$



$$\Rightarrow \left(\frac{x^2}{2}-\frac{x^4}{4}\right)_0^{\alpha} = \frac{1}{2}\left(\frac{x^2}{2}-\frac{x^4}{4}\right)_0^{1}$$

$$\Rightarrow \frac{\alpha^2}{2}-\frac{\alpha^4}{4}=\frac{1}{2}\left(\frac{1}{2}-\frac{1}{4}\right)$$

$4\alpha^2-2\alpha^4=1$

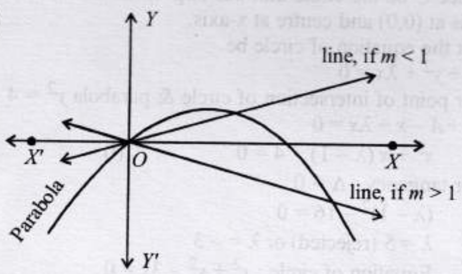
$\Rightarrow 2\alpha^4-4\alpha^2+1=0$

$$\Rightarrow \alpha^2=\frac{4\pm\sqrt{16-8}}{4}=1\pm\frac{1}{\sqrt{2}}$$

$$\therefore \quad 0<\alpha<1 \quad \Rightarrow \quad \alpha^2=1-\frac{1}{\sqrt{2}}$$

$$\Rightarrow \sqrt{0.29} > \sqrt{0.25} = \frac{1}{2} \text{ also } \alpha<1 \Rightarrow \frac{1}{2}<\alpha<1$$

20. **(b, d)**



The given curve is $y=x-x^2$ and $y=mx$

The two curves meet at

$mx=x-x^2$ or $x^2=x(1-m)$, $\therefore$ $x=0, 1-m$

The region bounded by curves

$$=\int_{P}(y_1-y_2)\,dx=\int_{L}(x-x^2-mx)\,dx$$

Clearly $m<1$ or $m>1$, but $m\neq 1$

$$\text{Now, } \int_0^{1-m}(1-x-x^2)\,dx=\left[(1-m)\frac{x^2}{2}-\frac{x^3}{3}\right]_0^{1-m}$$

$$=\frac{9}{2}, \text{if } m<1$$

or $(1-m)^3=27\Rightarrow 1-m=-3$, $\therefore$ $m=-2$

But if $m>1$ then $1-m$ is – ive, then $\int_{1-m}^{0}(1-x-x^2)\,dx$

$$=\left[(1-m)\frac{x^2}{2}-\frac{x^3}{3}\right]_{1-m}^{0}=\frac{9}{2}$$

$\therefore$ $(1-m)^3=-27\Rightarrow 1-m=-3$, or $1-m=-3$, $\therefore$ $m=4$.

21. **(A) → s; (B) → s; (C) → p; (D) → r**

(A) $$\int_{-1}^{1}\frac{dx}{1+x^2}=[\tan^{-1}x]_{-1}^{1}=\tan^{-1}(1)-\tan^{-1}(-1)$$

$$=\frac{2\pi}{4}=\frac{\pi}{2}$$

(B) $$\int_0^1\frac{dx}{\sqrt{1-x^2}}=\left[\sin^{-1}x\right]_0^1=\sin^{-1}(1)-\sin^{-1}(0)$$

$$=\frac{\pi}{2}-0=\frac{\pi}{2}$$

(C) $$\int_2^3\frac{dx}{1-x^2}=\left[\frac{1}{2}\log\left|\frac{1+x}{1-x}\right|\right]_2^3=\frac{1}{2}[\log 2-\log 3]$$

$$=\frac{1}{2}\log 2/3$$

(D) $$\int_1^2\frac{dx}{x\sqrt{x^2-1}}=\left[\sec^{-1}x\right]_1^2=\sec^{-1}2-\sec^{-1}1$$

$$=\frac{\pi}{3}-0=\frac{\pi}{3}$$

22. **(c)** $f(x)=4x^3+3x^2+2x+1$

$\because$ $f(x)$ is a cubic polynomial

$\therefore$ It has at least one real root.

Also $f'(x)=12x^2+6x+2=2\left(6x^2+3x+1\right)>0\ \forall\ x\in R$

$\therefore$ $f(x)$ is strictly increasing function

$\Rightarrow$ There is only one real root of $f(x)=0$

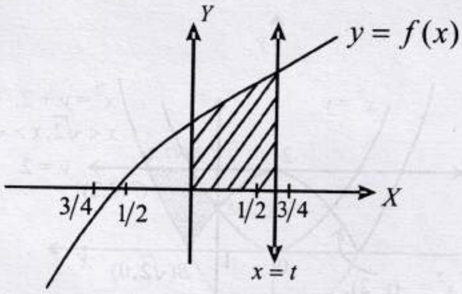
$$\text{Also } f(-1/2)=1-1+\frac{3}{4}-\frac{1}{2}>0$$

$$\text{and } f(-3/4)=1-\frac{3}{2}+\frac{27}{16}-\frac{27}{16}<0$$

$\therefore$ Real root lies between $-\frac{3}{4}$ and $-\frac{1}{2}$ and hence

$$s\in\left(-\frac{3}{4},-\frac{1}{2}\right)$$

23. **(a)** $y=f(x), x=0, y=0$ and $x=t$ bounds the area as shown in the figure



$\therefore$ Required area is given by

$$A = \int_0^t dx = \int_0^t \left(4x^3 + 3x^2 + 2x + 1\right) dx$$

$$= t^4 + t^3 + t^2 + t = t(t+1)\left(t^2+1\right)$$

New $\frac{1}{2} < t < \frac{3}{4}$; $\frac{3}{2} < t+1 < \frac{7}{4}$; $\frac{5}{4} < t^2 + 1 < \frac{25}{16}$

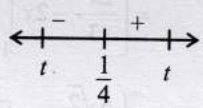
$$\therefore \frac{1}{2} \times \frac{3}{2} \times \frac{5}{4} < A < \frac{3}{4} \times \frac{7}{4} \times \frac{25}{16}$$

or $A \in \left(\frac{15}{16}, \frac{525}{256}\right) \subset \left(\frac{3}{4}, 3\right)$

24. **(b)** $f'(x) = 2(6x^2 + 3x + 1)$

$f''(x) = 6(4x+1) \Rightarrow$ Critical point $x = -1/4$

$\therefore$ decreasing in $\left(-t, -\frac{1}{4}\right)$



and increasing in $\left(-\frac{1}{4}, t\right)$

25. We have, $\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix} \begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2 + 3a \\ 3b^2 + 3b \\ 3c^2 + 3c \end{bmatrix}$

$\Rightarrow 4a^2 f(-1) + 4af(1) + f(2) = 3a^2 + 3a$

$4b^2 f(-1) + 4bf(1) + f(2) = 3b^2 + 3a$

$4c^2 f(-1) + 4cf(1) + f(2) = 3c^2 + 3c$

Consider the equation

$4x^2 f(-1) + 4xf(1) + f(2) = 3x^2 + 3x$

or $[4f(-1) - 3]x^2 + [4f(1) - 3]x + f(2) = 0$

Then clearly this eqn. is satisfied by x = a,b,c

A quadratic eqn. satisfied by more than two values of x means it is an identity and hence

$4f(-1) - 3 = 0 \quad \Rightarrow f(-1) = 3/4$

$4f(1) - 3 = 0 \quad f(1) = 3/4$

$f(2) = 0 \quad f(2) = 0$

Let $f(x) = px^2 + qx + r$ [$f(x)$ being a quadratic eqn.]

$f(-1) = \frac{3}{4} \Rightarrow p - q + r = \frac{3}{4}$

$f(1) = \frac{3}{4} \Rightarrow p + q + r = \frac{3}{4}$

$f(2) = 0 \Rightarrow 4p + 2q + r = 0$

Solving the above we get $q = 0$, $p = -\frac{1}{4}$, $r = 1$

$\therefore \quad f(x) = -\frac{1}{4}x^2 + 1$

It's maximum value occur at $f'(x) = 0$

i.e., $x = 0$ then $f(x) = 1$, $\therefore V(0, 1)$

Let A (–2, 0) be the point where curve meet x-axis.

Let B be the point $\left(h, \frac{4-h^2}{4}\right)$

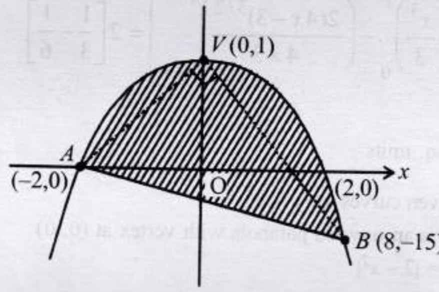
As $\angle AVB = 90^0$, $m_{AV} \times m_{BV} = -1$

$$\Rightarrow \left(\frac{0-1}{-2-1}\right) \times \left(\frac{\frac{4-h^2}{4} - 1}{h - 0}\right) = -1 \Rightarrow h = 8$$

$\therefore \quad B(8, -15)$

Equation of chord AB is

$$y + 15 = \frac{0-(-15)}{-2-8}(x-8) \Rightarrow 3x + 2y + 6 = 0$$



Required area is the area of shaded region given by

$$= \int_{-2}^{2}\left(-\frac{x^2}{4} + 1\right)dx + \int_{-2}^{8}\left\{-\left(\frac{-6-3x}{2}\right)\right\}dx - \int_{2}^{8}\left\{-\left(-\frac{x^2}{4} + 1\right)\right\}dx$$

$$= 2\int_0^2\left(-\frac{x^2}{4} + 1\right)dx + \frac{1}{2}\int_{-2}^{8}(6+3x)\,dx + \frac{1}{4}\int_2^8\left(-x^2 + 4\right)dx$$

$$= 2\left[\left(\frac{-x^3}{12} + x\right)_0^2\right] + \frac{1}{2}\left[6x + \frac{3x^2}{2}\right]_{-2}^{8} + \frac{1}{4}\left[\frac{-x^3}{3} + 4x\right]_2^8$$

$$= 2\left[\frac{-8}{12} + 2\right] + \frac{1}{2}[(48 + 3 \times 32) - (-12 + 6)]$$

$$+ \left[\frac{1}{4}\left(\frac{-512}{3} + 32\right) - \left(\frac{-8}{3} + 8\right)\right] = \frac{125}{3} \text{ sq. units.}$$

26. The given curves are, $x^2 = y$ ....(i)

$x^2 = -y$ ....(ii)

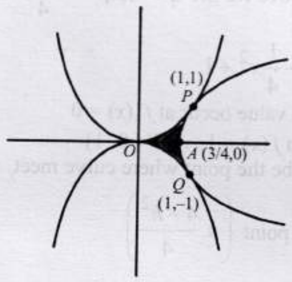
$y^2 = 4x - 3$ ....(iii)

Clearly point of intersection of (i) and (ii) is (0, 0). For point of intersection of (i) and (iii), solving them as follows

$x^4 - 4x + 3 = 0\ (x-1)(x^3 + x^2 + x - 3) = 0$

or $(x-1)^2(x^2 + 2x + 3) = 0$; $\Rightarrow x = 1$ and then $y = 1$

$\therefore$ Req. point is (1, 1). Similarly point of intersection of (ii) and (iii) is (1, – 1). The graph of three curves is as follows:



We also observe that at $x = 1$ and $y = 1$

$\frac{dy}{dx}$ for (i) and (iii) is same and hence the two curves touch each other at (1, 1).

Same is the case with (ii) and (iii) at (1, –1).

Required area = Shaded region in figure = 2 (*Ar OPA*)

$$= 2\left[\int_0^1 x^2 dx - \int_{3/4}^1 \sqrt{4x-3}\ dx\right]$$

$$= 2\left[\left(\frac{x^3}{3}\right)_0^1 - \left(\frac{2(4x-3)^{3/2}}{4\times 3}\right)_{3/4}^1\right] = 2\left[\frac{1}{3} - \frac{1}{6}\right]$$

$= \frac{1}{3}$ sq. units

27. The given curves are $y = x^2$

which is an upward parabola with vertex at (0, 0)

$y = |2 - x^2|$

or $y = \begin{cases} 2 - x^2 \text{ if } & -\sqrt{2} \le x \le \sqrt{2} \\ x^2 - 2 \text{ if } x < -\sqrt{2} \text{ or } x > \sqrt{2} \end{cases}$

or $x^2 = -(y-2)$; $-\sqrt{2} < x < \sqrt{2}$

a downward parabola with vertex at (0, 2)

$x^2 = y + 2$; $x < -\sqrt{2}$, $x > \sqrt{2}$

An upward parabola with vertex at (0, – 2)

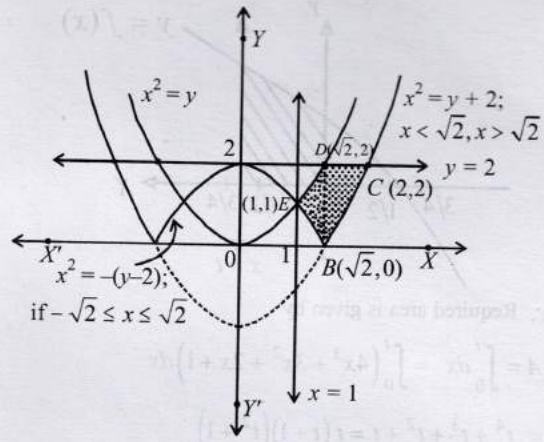
$y = 2$

A straight line parallel to x – axis

$x = 1$

A straight line parallel to y – axis

The graph of these curves is as follows.



$\therefore$ Required area = *BCDEB*

$$= \int_1^{\sqrt{2}} [Y_{(1)} - Y_{(2)}]dx + \int_{\sqrt{2}}^2 [Y_{(4)} - Y_{(3)}]dx$$

$$= \int_1^{\sqrt{2}} [x^2 - (2 - x^2)]dx + \int_{\sqrt{2}}^2 [2 - (x^2 - 2)]dx$$

$$= \int_1^{\sqrt{2}} (2x^2 - 2)\,dx + \int_{\sqrt{2}}^2 (4 - x^2)dx$$

$$= \left[\frac{2x^3}{3} - 2x\right]_1^{\sqrt{2}} + \left[4x - \frac{x^3}{3}\right]_{\sqrt{2}}^2$$

$$= \left(\frac{4\sqrt{2}}{3} - 2\sqrt{2} - \frac{2}{3} + 2\right) + \left(8 - \frac{8}{3} - 4\sqrt{2} + \frac{2\sqrt{2}}{3}\right)$$

$$= \left(\frac{20}{3} - 4\sqrt{2}\right) \text{ sq. units.}$$

28. $f(x) = \begin{cases} x^2 + ax + b; \ x < -1 \\ 2x \qquad ; -1 \le x \le 1 \\ x^2 + ax + b; \ x > 1 \end{cases}$

$\because f(x)$ is continuous at $x = -1$ and $x = 1$

$\therefore (-1)^2 + a(-1) + b = -2$

and $2 = (1)^2 + a \,.\, 1 + b$

i.e. $a - b = 3$ and $a + b = 1$

On solving we get $a = 2$, $b = -1$

$\therefore\ f(x) = \begin{cases} x^2 + 2x - 1; \ x < -1 \\ 2x \qquad ; -1 \le x \le 1 \\ x^2 + 2x - 1; \ x > 1 \end{cases}$

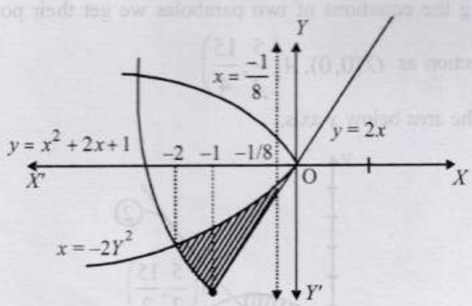
Given curves are $y = f(x)$, $x = -2y^2$ and $8x + 1 = 0$

Solving $x = -2y^2$, $y = x^2 + 2x - 1$ $(x < -1)$ we get $x = -2$

Also $y = 2x$, $x = -2y^2$ meet at (0, 0)

$y = 2x$ and $x = -1/8$ meet at $\left(-\frac{1}{8}, \frac{-1}{4}\right)$

The required area is the shaded region in the figure.



$\therefore$ Required area

$$=\int_{-2}^{-1}\left[\sqrt{\frac{-x}{2}}-(x^2+2x-1)\right]dx+\int_{-1}^{-1/8}\left[\sqrt{\frac{-x}{2}}-2x\right]dx$$

$$=\left[\frac{1}{\sqrt{2}}\frac{2(-x)^{3/2}}{3}-\frac{x^3}{3}-x^2+x\right]_{-2}^{-1}+\left[\frac{1}{\sqrt{2}}\frac{2(-x)^{3/2}}{3}-x^2\right]_{-1}^{-1/8}$$

$$=\left(\frac{\sqrt{2}}{3}+\frac{1}{3}-1-1\right)-\left(\frac{4}{3}+\frac{8}{3}-4-2\right)$$

$$+\left(\frac{\sqrt{2}}{3}\cdot\frac{1}{16\sqrt{2}}-\frac{1}{64}\right)-\left(\frac{\sqrt{2}}{3}-1\right)=\frac{257}{192}\text{ sq. units}$$

29. $f(x)=x^3-x^2$

Let P be on C_1, $y=x^2$ be (t,t^2)

$\therefore$ ordinate of Q is also t^2.

Now Q lies on $y=2x$, and $y=t^2$

$\therefore$ $x=t^2/2$ $\therefore$ $Q\left(\frac{t^2}{2},t^2\right)$

For point R, $x=t$ and it is on $y=f(x)$

$\therefore$ R is $[t,f(t)]$

Area OPQ $=\int_0^{t^2}(\underset{C_1}{x_1}-\underset{C_2}{x_2})dy=\int_0^{t^2}\left(\sqrt{y}-\frac{y}{2}\right)dy$

$$=\frac{2}{3}t^3-\frac{t^4}{4}\qquad\text{...(i)}$$

Area $OPR=\int_{0_{C_1}}^{t}ydx+\left|\int_{0_{C_3}}^{t}ydx\right|$

$$=\int_0^t x^2dx+\left|\int_0^t f(x)dx\right|=\frac{t^3}{3}+\left|\int_0^t f(x)dx\right|\qquad\text{....(ii)}$$

Equating (i) and (ii), we get,

$$\frac{t^3}{3}-\frac{t^4}{4}\left|\int_0^t f(x)dx\right|$$

Differentiating both sides,we get,

$$t^2-t^3=-f(t)$$

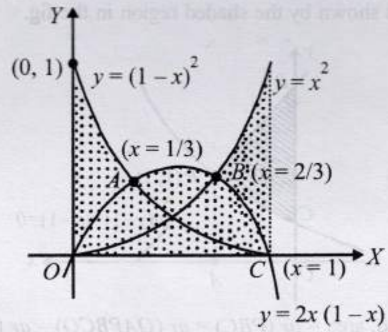
$\therefore$ $f(x)=x^3-x^2$.

30. We draw the graph of $y=x^2$, $y=(1-x)^2$ and $y=2x(1-x)$ in figure. Let us find the point of intersection of $y=x^2$ and $y=2x(1-x)$ The x – coordinate of the point of intersection satisfies the equation $x^2=2x(1-x)$, $\Rightarrow 3x^2=2x \Rightarrow 0$ or $x=2/3$

$\therefore$ At B, $x=2/3$

Similarly, we find the x coordinate of the points of intersection of $y=(1-x)^2$ and $y=2x(1-x)$ are $x=1/3$ and $x=1$

$\therefore$ At A, $x=1/3$ and at C, $x=1$



From the figure it is clear that

$$f(x)=\begin{cases}(1-x)^2 & \text{for } 0\le x\le 1/3\\ 2x(1-x) & \text{for } 1/3\le x\le 2/3\\ x^2 & \text{for } 2/3\le x\le 1\end{cases}$$

The required area A is given by

$$A=\int_0^1 f(x)dx$$

$$=\int_0^{1/3}(1-x)^2dx+\int_{1/3}^{2/3}2x(1-x)dx+\int_{2/3}^{1}x^2dx$$

$$=\left[-\frac{1}{3}(1-x)^3\right]_0^{1/3}+\left[x^2-\frac{2x^2}{3}\right]_{1/3}^{2/3}+\left[\frac{1}{3}x^3\right]_{2/3}^{1}$$

$$=\frac{17}{27}\text{ sq. units}$$

31. The normal to the curve at $P(1,1)$ is

$a(y-1)+(x-1)=0$

First we consider the case when $a\neq 0$

Slope of normal at $P(1,1)$ $=-\frac{1}{a}$

$\Rightarrow$ Slope of the tangent at $(1,1)$ is $=a$

$$\Rightarrow\left(\frac{dy}{dx}\right)_{(1,1)}=a\qquad\text{... (i)}$$

But we are given that

$$\frac{dy}{dx}\propto y\Rightarrow\frac{dy}{dx}=ky\Rightarrow\frac{dy}{y}=kdx$$

$\Rightarrow \log|y|=kx+c \Rightarrow |y|=e^{kx+c}=e^c.e^{kx}$

$\Rightarrow y=\pm e^c e^{kx} \Rightarrow y=Ae^{kx}$, where A is constant.

Since the curve passes through $(1, 1)$,

$\therefore$ $1=Ae^k \Rightarrow A=e^{-k}$

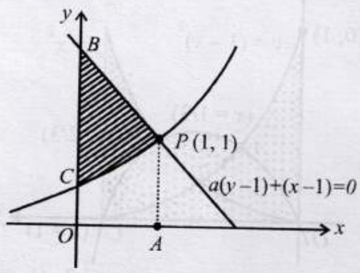
$\therefore$ $y=e^{k(x-1)}$ M$\Rightarrow \frac{dy}{dx}=ke^{k(x-1)}$

$$\Rightarrow\left(\frac{dy}{dx}\right)_{(1,1)}=k\qquad\text{...(ii)}$$

Now, from (i) and (ii), $\left(\frac{dy}{dx}\right)_{1,1} = a = k$

$\therefore \quad y = e^{a(x-1)}$ which is the required curve.

Now the area bounded by the curve, y-axis and normal to curve at (1, 1) is shown by the shaded region in the fig.



$\therefore$ Req. area = $ar\,(PBC) = ar\,(OAPBCO) - ar\,(OAPCO)$

$$= \int_0^1 y_{\text{normal}}dx - \int_0^1 y_{\text{curve}}dx$$

$$= \int_0^1\left(-\frac{1}{a}(x-1)+1\right)dx - \int_0^1 e^{a(x-1)}dx$$

$$= \left[-\frac{1}{2a}(x-1)^2 + x\right]_0^1 - \left[\frac{1}{a}e^{a(x-1)}\right]_0^1$$

$$= 1 + \frac{1}{2a} - \frac{1}{a} + \frac{1}{a}e^{-a} = 1 + \frac{1}{a}e^{-a} - \frac{1}{2a}$$

Now we consider the case when $a = 0$. Then normal at (1, 1) becomes $x - 1 = 0$, which is parallel to y-axis, therefore tangent at (1, 1) should be parallel to x-axis.

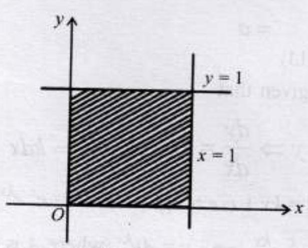
$$\therefore \quad \left(\frac{dy}{dx}\right)_{(1,1)} = 0 \qquad \text{...(iii)}$$

$\because \quad \frac{dy}{dx} \propto y$ and $y = e^{k(x-1)}$ (as in $a \neq 0$ case)

$$\therefore \quad \frac{dy}{dx} = ke^{k(x-1)}$$

$$\left(\frac{dy}{dx}\right)_{(1,1)} = k \qquad \text{...(iv)}$$

From (iii) and (iv), we get $k = 0$ and required curve becomes $y = 1$



In this case the required area
= shaded area in fig. = 1 sq. unit.

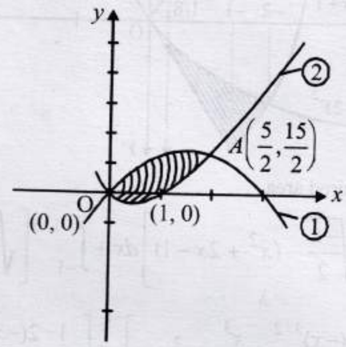
32. The given equations of parabola are

$$y = 4x - x^2 \text{ or } (x-2)^2 = -(y-4) \qquad \text{...... (i)}$$

$$\text{and } y = x^2 - x \text{ or } \left(x - \frac{1}{2}\right)^2 = \left(y + \frac{1}{4}\right) \qquad \text{...... (ii)}$$

Solving the equations of two parabolas we get their points of intersection as $O(0,0), A\left(\frac{5}{2}, \frac{15}{4}\right)$

Here the area below x-axis,



$$A_1 = \int_0^1 (-y_2)\,dx = \int_0^1 (x - x^2)dx$$

$$= \left(\frac{x^2}{2} - \frac{x^3}{3}\right)_0^1 = \frac{1}{6} \text{ sq. units.}$$

Area above x-axis,

$$A_2 = \int_0^{5/2}(4x - x^2)\,dx - \int_1^{5/2}(x^2 - x)\,dx$$

$$= \left(2x^2 - \frac{x^3}{3}\right)_0^{5/2} - \left(\frac{x^3}{3} - \frac{x^2}{2}\right)_1^{5/2}$$

$$= \left(\frac{25}{2} - \frac{125}{24}\right) - \left[\left(\frac{125}{24} - \frac{25}{8}\right) - \left(\frac{1}{3} - \frac{1}{2}\right)\right] = \frac{121}{24}$$

$\therefore$ Ratio of areas above x- axis and below x – axis.

$$A_2 : A_1 = \frac{121}{24} : \frac{1}{6} = \frac{121}{4} = 121 : 4$$

33. The given curves are $y = x^2$ and $y = \frac{2}{1+x^2}$. The curve $y = x^2$ is upward parabola with vertex at origin.

Also, $y = \frac{2}{1+x^2}$ is a curve symm. with respect to y-axis.

At $x = 0, y = 2$,

$$\frac{dy}{dx} = \frac{-4x}{(1+x^2)^2} < 0 \quad \text{for } x > 0$$

$\therefore$ Curve is decreasing on $(0, \infty)$

Moreover $\frac{dy}{dx} = 0$ at $x = 0$

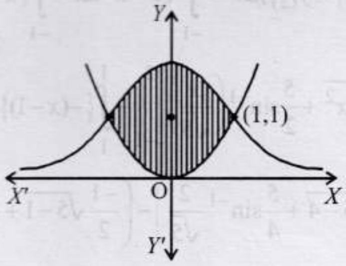
$\Rightarrow$ At (0,2) tangent to curve is parallel to x-axis.

Since $x \to \infty$, $y \to 0$

$\therefore$ y = 0 is asymptote of the given curve.

For the given curves, point of intersection : solving their equations we get $x = 1, y = 1$, i.e., (1,1).

Thus the graph of two curves is as follows:



$\therefore$ The required area $= 2\int_0^1\left(\frac{2}{1+x^2}-x^2\right)dx$

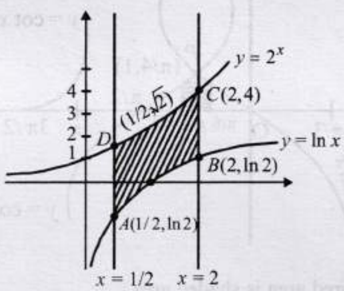
$$= \left(4\tan^{-1}x-\frac{2x^3}{3}\right)_0^1 = \pi-\frac{2}{3} \text{ sq. units.}$$

34. The given curves are

$x=\frac{1}{2}$ ... (i), $x=2$...(ii), $y=\ln x$...(iii), $y=2^x$ ...(iv)

Line $x=\frac{1}{2}$ meets the curve (3) at $\left(\frac{1}{2},-\ln 2\right)$ and (4) at $\left(\frac{1}{2},\sqrt{2}\right)$. Line $x=2$ meets the curve (3) at $(2, \ln 2)$ and (4) at $(2, 4)$.

The graph of curves are as shown in the figure.



Required area = ABCDA

$$= \int_{1/2}^{1}(-\ln x)dx+\int_{1/2}^{2}2^x dx-\int_{1}^{2}\ln x\, dx$$

$$= \int_{1/2}^{2}2^x dx-\int_{1/2}^{2}\ln x\, dx = \int_{1/2}^{2}(2^x-\ln x)\,dx$$

$$= \left(\frac{4-\sqrt{2}}{\log 2}-\frac{5}{2}\log 2+\frac{3}{2}\right)$$

35. $y=x(x-1)^2,\ 0\le x\le 2$

$$\frac{dy}{dx}=(x-1)^2+2x(x-1)=(x-1)(3x-1)$$

For max. or min., put $\frac{dy}{dx}=0$

$\Rightarrow (x-1)(3x-1)=0 \Rightarrow x=1, 1/3$

$$\frac{d^2y}{dx^2}=3x-1+3(x-1)=6x-4$$

At $x=1$, $\frac{d^2y}{dx^2}=2(+ve)$, $\therefore$ y is min. at $x=1$

At $x=1/3$, $\frac{d^2y}{dx^2}=-2(-ve)$, $\therefore$ y is max. at $x=1/3$

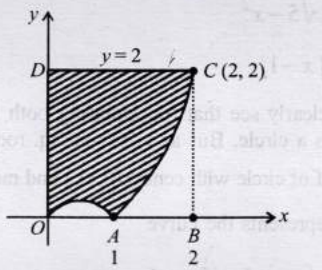
$\therefore$ Max value of y is $=\frac{1}{3}\left(\frac{1}{3}-1\right)^2=\frac{4}{27}$

Min. value of y is $= 1\ (1-1)^2 = 0$

Now the curve cuts the axis x at $(0, 0)$ and $(1, 0)$. When x increases from 1 to 2, y also increases and is +ve.

When $y=2$, $x(x-1)^2=2 \Rightarrow x=2$

Using max./min. values of y and points of intersection with x-axis, we get the curve as in figure and shaded area is the required area.



$\therefore$ The required area = Area of square $OBCD-\int_0^2 y\,dx$

$$= 2\times 2-\int_0^2 x(x-1)^2\,dx$$

$$= 4-\left[\left(x\frac{(x-1)^3}{3}\right)_0^2-\frac{1}{3}\int_0^2(x-1)^3.1dx\right]$$

$$= 4-\left[\frac{x}{3}(x-1)^3-\frac{(x-1)^4}{12}\right]_0^2=\frac{10}{3}\text{sq.units.}$$

36. We have to find the area bounded by the curves

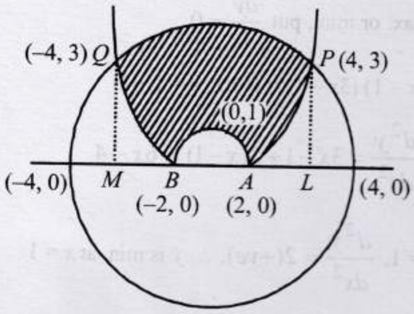
$x^2+y^2=25$ ...(i)

$4y=|4-x^2|$ ...(ii)

$x=0$ ...(iii)

So, points of intersection of parabola with x-axis are $(2, 0)$ and $(-2, 0)$

The points of intersection of (1) and (2) are $(4, 3)$ and $(-4, 3)$



Required area is

$$= 2\left[\int_0^4 y_{circle}dx - \int_0^2 y_{P_1}dx - \int_2^4 y_{P_2}dx\right]$$

$$= 2\left[\int_0^4 \sqrt{25-x^2}\,dx - \frac{1}{4}\int_0^2 (4-x^2)dx - \frac{1}{4}\int_0^4 (x^2-4)dx\right]$$

$$= 2\left[\left[\frac{x}{2}\sqrt{25-x^2} + \frac{25}{2}\sin^{-1}\frac{x}{5}\right]_0^4 - \frac{1}{4}\left(4x - \frac{x^3}{3}\right)_0^2 - \frac{1}{4}\left(\frac{x^3}{3} - 4x\right)_2^4\right]$$

$$= 12 + 25\sin^{-1}\frac{4}{5} - 8 = 4 + 25\sin^{-1}\frac{4}{5}$$

37. The given curves are

$$y = \sqrt{5-x^2} \quad \text{... (i)}$$

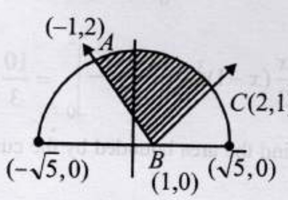
$$y = |x-1| \quad \text{... (ii)}$$

We can clearly see that (on squaring both sides of (1)) eq. (i) represents a circle. But as y is + ve sq. root, $\therefore$ (1) represents upper half of circle with centre (0, 0) and radius $\sqrt{5}$.

Eq. (ii) represents the curve

$$y = \begin{cases} -x+1 & \text{if } x < 1 \\ x-1 & \text{if } x \geq 1 \end{cases}$$

Graph of these curves are as shown in figure with point of intersection of $y = \sqrt{5-x^2}$ and $y = -x+1$ as $A(-1,2)$ and of $y = \sqrt{5-x^2}$ and $y = x-1$ as $C(2,1)$.



The required area = Shaded area

$$= \int_{-1}^{2} (y_{(1)} - y_{(2)})dx = \int_{-1}^{2} \sqrt{5-x^2}\,dx - \int_{-1}^{2} |x-1|\,dx$$

$$= \left[\frac{x}{2}\sqrt{5-x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)\right]_{-1}^{2} - \int_{-1}^{1}\{-(x-1)\}\,dx - \int_{1}^{2}(x-1)dx$$

$$= \left(\frac{2}{2}\sqrt{5-4} + \frac{5}{4}\sin^{-1}\frac{2}{\sqrt{5}}\right) - \left(\frac{-1}{2}\sqrt{5-1} + \frac{5}{2}\sin{-1}\left(\frac{-1}{\sqrt{5}}\right)\right)$$

$$- \left(\frac{-x^2}{2} + x\right)_{-1}^{1} - \left(\frac{x^2}{2} - x\right)_{1}^{2}$$

$$= 2 + \frac{5}{2}\left[\sin^{-1}\frac{2}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{5}}\right] - 2 - \frac{1}{2}$$

$$= \frac{5}{2}\left[\sin^{-1}\frac{2}{\sqrt{5}} + \cos^{-1}\frac{2}{\sqrt{5}}\right] - \frac{1}{2} = \frac{5}{2}\left(\frac{\pi}{2}\right) - \frac{1}{2}$$

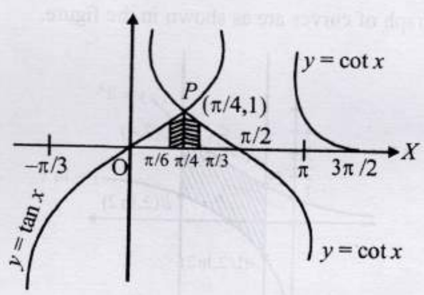
$$= \frac{5\pi - 2}{4} \text{ square units.}$$

38. To find the area bold by x - axis and curves

$$y = \tan x, -\pi/3 \leq x \leq \pi/3 \quad \text{... (i)}$$

and $y = \cot x, \pi/6 \leq x \leq 3\pi/2$ ... (ii)

The curves intersect at P, where $\tan x = \cot x$, which is satisfied at $x = \pi/4$ within the given domain of x.



The required area is shaded area

$$A = \int_{\pi/6}^{\pi/4} \tan x\,dx + \int_{\pi/4}^{\pi/3} \cot x\,dx$$

$$= [\log \sec x]_{\pi/6}^{\pi/4} + [\log \sin x]_{\pi/4}^{\pi/3}$$

$$= 2\left(\log \sqrt{2}.\frac{\sqrt{3}}{2}\right) = 2\log\sqrt{\frac{3}{2}} = \log 3/2 \text{ sq. units}$$